

Supplemental notes on gear ratios, torque and speed

Vocabulary

- SI (Système International d'Unités) – the metric system
- force
- torque
- axis
- moment arm
- acceleration
- gear ratio
- newton – Si unit of force
- meter – SI unit of distance
- newton-meter – SI unit of torque

Torque

Torque is a measure of the tendency of a force to rotate an object about some axis. A torque is meaningful only in relation to a particular axis, so we speak of the torque *about the motor shaft*, the torque *about the axle*, and so on. In order to produce torque, the force must act at some distance from the axis or pivot point. For example, a force applied at the end of a wrench handle to turn a nut around a screw located in the jaw at the other end of the wrench produces a torque about the screw. Similarly, a force applied at the circumference of a gear attached to an axle produces a torque about the axle. The perpendicular distance d from the line of force to the axis is called the moment arm.

In the following diagram, the circle represents a gear of radius d . The dot in the center represents the axle (A). A force F is applied at the edge of the gear, tangentially.

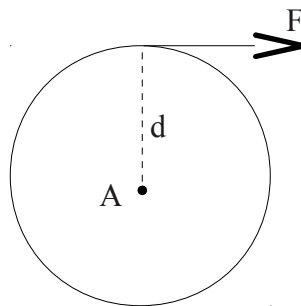


Diagram 1

In this example, the radius of the gear is the moment arm. The force is acting along a tangent to the gear, so it is perpendicular to the radius. The amount of torque at A *about the gear axle* is defined as

$$\tau = F \times d$$

We use the Greek letter Tau (τ) to represent torque. The SI (metric) unit of force is the *newton*, and the unit of distance is *meters*. Since torque is the product of force times distance, the unit of torque is *newton-meters*. (Do NOT read this as ~~newtons per meter~~, which would indicate division, but the hyphenated term *newton-meters*, or better still, *newton·meters*, indicating that it is a product.)

So, given a force and a moment arm (distance), we can use the above formula to compute the resulting torque. For example, referring to Diagram 1, if we are given that the force F is 20 *newtons* and the radius d is 3 centimeters (0.03 *meters*), we can calculate the torque about the axle as

$$\tau = 20 \text{ newtons} \times 0.03 \text{ meters} = 0.6 \text{ newton} \cdot \text{meters}$$

Conversely, if we already know the torque acting about the axle of a wheel and we know the radius, we can calculate the force that is applied along a tangent to the edge of the wheel by dividing the torque by the moment arm. This is useful because it enables us to determine the horizontal force that the wheel applies to the floor, pushing the robot forward.

$$F = \frac{\tau}{d}$$

For example, referring again to Diagram 1, if we know that a torque of 0.54 *newton-meters* is applied by a motor about the axle at A, and that the radius d is 3 centimeters, then we can calculate the force applied at the edge, tangential to the wheel or gear, as

$$F = \frac{0.54 \text{ newton} \cdot \text{meters}}{0.03 \text{ meters}} = 18 \text{ newtons}$$

Acceleration

Why is it useful to know the force applied at the edge of a wheel? Because it gives us information about how rapidly the wheel (and the vehicle or robot attached to the wheel) will accelerate. Newton's Second Law states that the acceleration of an object is directly proportional to the net force acting on it, and inversely proportional to its mass. This can be expressed by the equation

$$a = \frac{F}{m}$$

The greater the force, the faster the object will accelerate. If we double the force, the rate of acceleration doubles, and so on.

Caution: Acceleration is not the same thing as speed. Acceleration is the *rate of change of speed*. It is the rate at which the speed of an object increases. Negative acceleration (deceleration) is the rate at which the speed of an object decreases. In the metric system, a common unit of speed is *kilometers/second*. Accordingly, the unit of acceleration is *kilometers/second/second* or *km/sec²*. Read that as “kilometers per second squared”.

Note that this does NOT tell us how fast the object will move; it does let us determine how quickly the object gets up to a particular speed.

Gear Ratios and Torque

When a series of gears is used to transmit power from a motor to a wheel, the gear connected to the motor is called the driver or input gear, and the gear connected to the wheel is called the driven or output gear. In general, gears situated in between the driver and the driven gears are called idler gears.

The gear ratio is the ratio of the **number of teeth on the output gear** (the one connected to the wheel) to the **number of teeth on the input gear** (connected to the motor).

Remember -- the gear ratio is the ratio of
output : input (read this as "output to input")
driven : driver (read as "driven to driver")

Since a ratio is just another way to express a fraction, we can also write the gear ratio as:

$$GR = \frac{\text{Output}}{\text{Input}} = \frac{\text{Driven}}{\text{Driver}}$$

Equivalently, it is the ratio of the circumference of the output gear to the circumference of the input gear, because the number of teeth on each gear is proportional to the gear's circumference C . Also, since the formula for circumference is $C = \pi D$ and diameter (D) is twice the radius (R) we can write

$$GR = \frac{\pi D_o}{\pi D_i} = \frac{D_o}{D_i} = \frac{2R_o}{2R_i} = \frac{R_o}{R_i}$$

(The subscripts o and i refer to the output and input gears respectively.)

The gear ratio expresses the ratio of the output torque to the input torque. Thus, we can multiply the torque supplied at the motor shaft (the input) by the gear ratio to find the torque at the wheel axle (the output).

We can calculate the torque at the wheel axle as follows:

$$\text{WheelTorque} = \text{MotorTorque} \times \frac{\text{OutputSize}}{\text{InputSize}}$$

or more simply:

$$\text{WheelTorque} = \text{MotorTorque} \times GR$$

For example, if the torque about the motor shaft is 8 *newtons* (newton is the metric unit for torque), the gear attached to the motor shaft has 16 teeth and the gear attached to the wheel axle has 48 teeth, the torque at the wheel axle is

$$\text{WheelTorque} = 8 \text{ newtons} \times \frac{48}{16} = 24 \text{ newtons}$$

That's fine if we know the torque at the motor shaft and want to find out the torque at the wheel axle. What if we know the torque at the wheel axle and want to find out the torque at the motor shaft? We can multiply both sides of the equation by the RECIPROCAL of the gear ratio:

$$\frac{InputSize}{OutputSize} \times WheelTorque = MotorTorque \times \frac{OutputSize}{InputSize} \times \frac{InputSize}{OutputSize}$$

Then, cancelling terms and exchanging the right and left sides of the equation gives us:

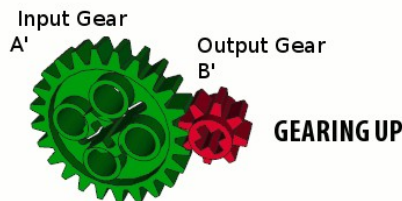
$$MotorTorque = WheelTorque \times \frac{InputSize}{OutputSize}$$

Gear Ratios and Speed

Transmitting power through a series of gears can also affect rotational speed. In a system consisting of only two gears, as they revolve each tooth of the input (driver) gear meshes with, and pushes, a tooth of the output (driven) gear. If the input gear has fewer teeth than the output gear, then the input gear will turn through a complete revolution sooner than the output gear. The output gear will rotate more slowly than the input. This is called *gearing down*. If the input gear has exactly *half* as many teeth as the output gear, the input gear will turn completely around in the same amount of time that the output gear takes to turn only half-way, so the output gear will rotate at half the speed of the input. Here is an illustration of gearing down.



On the other hand, if the input gear has more teeth than the output, the opposite occurs. In this case the output gear will rotate more quickly than the input. This is called *gearing up*. If the input gear has *twice* as many teeth as the output, the input gear will turn only half-way around in the time it takes for the output to turn completely around, so the output gear will rotate at twice the speed of the input. Here is an illustration of gearing up.



Given the number of teeth on both gears, if we know the rotational speed of the input gear, we can calculate the rotational speed of the output gear as follows:

$$OutputSpeed = InputSpeed \times \frac{InputSize}{OutputSize}$$

Since the input gear is connected directly to the motor shaft, it turns at the same rotational speed as the motor. Similarly, the output gear is connected directly (through the axle) to the wheel, so the wheel revolves at the same rotational speed as the output gear.

For example, if the motor rotates at 300 RPM, which means 300 *revolutions per minute* (300 *rev/min*), the input gear has 8 teeth and the output gear has 24 teeth (this is gearing down), the rotational speed of the wheel can be determined as follows:

$$\text{OutputRPM} = 300 \frac{\text{rev}}{\text{min}} \times \frac{8}{24} = 100 \frac{\text{rev}}{\text{min}}$$

On the other hand, if the motor rotates at 300 RPM, the input gear has 24 teeth and the output gear has 8 teeth (this is gearing up), the rotational speed of the wheel will be:

$$\text{OutputRPM} = 300 \frac{\text{rev}}{\text{min}} \times \frac{24}{8} = 900 \frac{\text{rev}}{\text{min}}$$

If there are one or more additional (idler) gears between the input and output gears, they can be disregarded in determining the final speed. It is sufficient to consider the relative sizes (or numbers of teeth) of the input and output gears.

Note that the *up* and *down* in gearing up and gearing down refers to the rotational speed, NOT torque. It is important to recognize that the fraction in the above equation (*InputSize/OutputSize*) is the **inverse** of the gear ratio, and so the effect of gearing on speed is the **inverse** of its effect on torque. That is, if as a result of the gear arrangement the torque increases, the rotational speed decreases. And if the torque decreases, the rotational speed increases.

Also, note that all references to speed in this section about gearing have been about **rotational** speed. This is the rate, in revolutions/minute, at which the components – wheels, gears, etc. -- rotate about their axes, and NOT the speed at which the robot travels along the floor.

Wheels and Speed

This final section will review how the wheel size affects the maximum speed at which the robot travels. Notice that term *maximum*. That is the maximum speed at which the robot will move along the floor. It assumes that there is sufficient torque to overcome the frictional forces that impede motion. This section does not address the question of how long it takes to *accelerate* the robot up to its maximum speed, or even if it can do so at all. That depends upon the force applied by the wheel horizontally along the floor, which in turn depends on the wheel size and the torque at the wheel axle, which in turn depends on the gearing and the amount of torque that the motor can produce. Here, we simply assume that the wheel has gotten up to “full speed”.

This calculation is simple. Remember that the circumference of the wheel is given by $C = \pi d$. As the wheel rotates along the floor, every point on the circumference of the wheel contacts a corresponding point on the floor. Imagine that you mark the point on a wheel that is in contact with the floor, also marking the floor at that point, then roll the wheel in a straight line until the original point is in contact with the floor again, and mark the floor again at that point. It is easy to see that the distance

between the two marks on the floor will be equal to the circumference of the wheel. Therefore we have an easy way to determine the distance that the robot travels for each rotation of its wheel: it is simply the circumference of the wheel. If we multiply the distance the robot travels during EACH rotation times the number of rotations per minute, we will know the distance travelled per minute. Therefore, the speed that the robot travels is the product of the circumference of its drive wheels (the wheels that are applying power to the floor) times the rotational speed of the wheels. In other words:

$$v = C \times \omega$$

In this equation, v represents the linear velocity (speed), C represents the circumference of the wheel, and ω (the Greek letter omega) represents the rotational speed.

For example, if a robot's drive wheel is 4 centimeters (4 *cm*) in diameter and it is rotating at 900 RPM (900 *rev/min*), the circumference of the wheel is

$$C = \pi d = 3.14 \times 4 \text{ cm} = 12.56 \text{ cm}$$

and the robot will travel along the floor at a speed of

$$v = C \times \omega = 12.56 \text{ cm} \times 900 \frac{\text{rev}}{\text{min}} = 11304 \frac{\text{cm}}{\text{min}}$$

We can translate this into more convenient units: 4 *cm* = 0.04 *m* and 900 *rev/min* = 15 *rev/sec* so

$$v = C \times \omega = \pi \times 0.04 \text{ m} \times 15 \frac{\text{rev}}{\text{sec}} = 1.884 \frac{\text{m}}{\text{sec}}$$